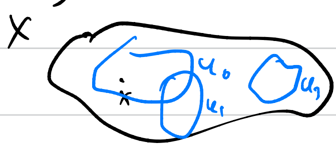
Ergodic Theory and Measured Group Theory
Lecture 9

Detour into compact realizations of a transformation.

Theorem. The shift transformation s on $(2^{\mathbb{N}})^{\mathbb{N}}$ is universal in the sense that every Borel transformation on a st. Borel space X equivariantly Borel embeds into the shift, i.e., $\exists$ Borel $\varphi : X \hookrightarrow (2^{\mathbb{N}})^{\mathbb{N}}$ s.t. $\varphi \circ T = s \circ \varphi$.

Proof. Let T be a Borel transformation on a st. Borel X. Even without using the Borel isom. theorem, one can show that $\exists$ Borel $X \hookrightarrow 2^{\mathbb{N}}$ as follows: Fix $(U_n)_{n\in\mathbb{N}}$ ctbl collection of Borel sets generating $\mathcal{B}(X)$ and map $X \hookrightarrow 2^{\mathbb{N}}$ by $x \mapsto (\mathbb{1}_{U_n}(x))_{n\in\mathbb{N}}$. This is easily seen to be Borel.



Thus, WLOG, $X := 2^{\mathbb{N}}$. Map each $x \in 2^{\mathbb{N}}$ to $(T^n x)_{n\in\mathbb{N}}$. This works. □

Cor. Any Borel transformation T on a st. prob. space (X,μ) is measure-isomorphic to the shift transformation on $(2^{\mathbb{N}})^{\mathbb{N}}$ with some Borel prob. meas.

Proof. Let $\varphi : X \hookrightarrow (2^{\mathbb{N}})^{\mathbb{N}}$ be a Borel embedding. Let $\nu := \varphi_* \mu$, i.e. $\nu(B) = \mu(\varphi^{-1}(B))$ $\forall B \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$. Then up to a ν-null set, φ is a Borel bijection, hence Borel isom. by the Luzin-Souslin thm from DST. $\square$

Countable groups (continued). ∘ Linear groups: – $GL_n(\mathbb{Z}) :=$ all invertible matrices with integer coefficients and $\neq 0$ determinant (hence $\det = \pm 1$), i.e. when acting on $\mathbb{R}^n$, these are volume-preserving transformations (i.e. measure-preserving).

– $SL_n(\mathbb{Z}) =$ all matrices in $GL_n(\mathbb{Q})$ but $\det = 1$, i.e. preserve orientation.

– $SO_n(\mathbb{Q}) :=$ norm-preserving matrices, i.e. orthogonal.

∘ Wreath products. Let A and Γ be ctbl groups. Denote:

$$A^{\Gamma}_{<\infty} := \bigoplus_{\Gamma} A := \{(a_\gamma)_{\gamma \in \Gamma} \in A^{\Gamma} : \operatorname{supp}(a_\gamma)_{\gamma\in\Gamma} \text{ is finite}\},$$

in particular, $A^{\Gamma}_{<\infty}$ is a ctbl group.

For example, $A := \mathbb{Z}/2\mathbb{Z}$ and $\Gamma := \mathbb{Z}$ then $A^{\mathbb{Z}}_{<\infty}$ is all finitely-supported binary bi-infinite sequences.

$\Gamma \curvearrowright A^{\Gamma}_{<\infty}$ by shift: $\gamma.(a_\delta)_{\delta\in\Gamma} := (a_{\delta\gamma})_{\delta\in\Gamma}$ (or $(a_{\gamma^{-1}\delta})_{\delta\in\Gamma}$, both are left actions). Thus $A^{\Gamma}_{<\infty} \rtimes \Gamma$ is defined by this shift action and is call the (restricted)

$\backslash\mathrm{wr}$

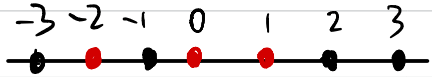
wreath product of A and Γ, and denoted by $A \wr \Gamma$.

For the example above of $A := \mathbb{Z}/2\mathbb{Z}$ and $\Gamma := \mathbb{Z}$, the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is called the **lamplighter group**. This is because for any $x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}_{<\infty}$ and $x_0 := 1_{\{0\}} \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$, i.e.

$$x_0 := \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \ldots \\ \ldots & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \end{array},$$

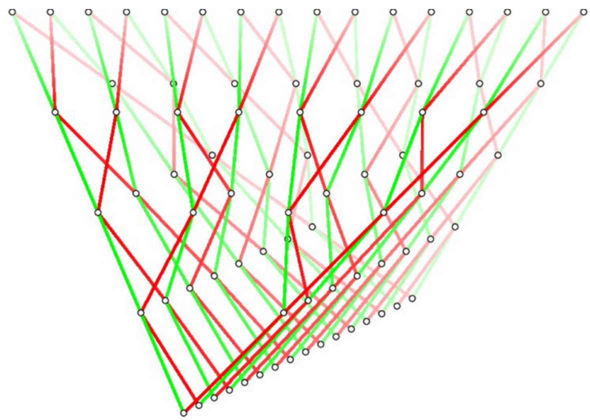
then

$$(x,1)\cdot(x_0,0) = (x+s(x_0),1) =$$



switches on/off.

The canonical generating set is $\{(x_0,0),(0,1)\}$. The corresponding Cayley graph is this:



Group actions.

- The rotation action of $\mathbb{Z}$ by some angle on S^1.
- The odometer action (homework) of $\mathbb{Z}$ on $2^{\mathbb{N}}$.
- The shift action of any ctbl gp Γ on X^Γ, where X is any st. Borel space, e.g. $X=2$. $\forall \gamma \in \Gamma$, $\gamma\cdot(x_\delta)_{\delta\in\Gamma} := (x_{\delta\gamma})_{\delta\in\Gamma}$.

If ν is a Borel prob. measure on X and $\mu := \nu^\Gamma$, then the shift action is μ-preserving. This action is called the Bernoulli action.

Remark. The shift action $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$ is universal among all Borel actions of Γ.

Def. $\Gamma \curvearrowright X$, where Γ is ctbl and X is st. Borel, is called a *Borel action* if each $\gamma \in \Gamma$ acts as a Borel (invertible) transformation.

More examples.

- $SO_n(\mathbb{Q}) \curvearrowright S^{n-1}$ by rotations is pmp, where the measure is the Lebesgue measure through the identification $S^{n-1} \cong [0,1)^{n-1}$.
- $GL_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$ by first acting on $\mathbb{R}^n$ as usual (matrix multiplication) then taking mod 1. This is measure-preserving because $\det = \pm 1$.
- Recall that $\mathbb{F}_d$, $d \leq \infty$, denotes the free group on d generators $S := \{a_1, a_2, \dots, a_d\}$. The *boundary* of $\mathbb{F}_d$ is $\partial \mathbb{F}_d :=$ all reduced infinite words, i.e.

sequences $(s_n)_{n\in\mathbb{N}}$, where $s_n \in S^{\pm 1}$, s.t. $s_n \neq s_{n+1}^{-1}$ $\forall n$.

Define the natural action of $\mathbb{F}_d$ on $\partial\mathbb{F}_d$ by concatenating and reducing, i.e. $\forall w \in \mathbb{F}_d$ and $x \in \partial\mathbb{F}_d$, $w \cdot x = \text{reduced}(wx)$. For example, for $\mathbb{F}_2 := \langle a, b \rangle$,

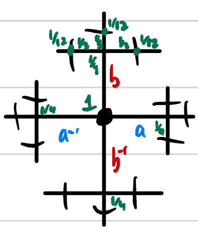
$$x := abba^{-1}ba^{-1}b^{-1}\ldots$$

$$ab^{-1} \cdot x = ab^{-1}x$$

$$b^{-1}a^{-1} \cdot x = ba^{-1}ba^{-1}b^{-1}\ldots$$

$$a^{-1} \cdot x = bbaa^{-1}ba^{-1}b^{-1}\ldots = s(x).$$

Let's define a measure on $\partial\mathbb{F}_2$. It's enough to define it on basic clopen sets $[w] := \{x \in \partial\mathbb{F}_2 : x \text{ begins with } w\}$, $w \in \mathbb{F}_2$. A prob. measure on $\partial\mathbb{F}_2$ can be given by a σ-finite measure on $\mathbb{F}_2$ s.t. $\sum_{w^\frown s \in \mathbb{F}_2} \text{weights}(w^\frown s) = \text{weight}(w)$, where s ranges over $S^{\pm 1}$.



For the first letter, we give $\frac{1}{4}$ prob. each, and each next letter $1/3$ prob. Thus, we get the uniform prob. on the set of all reduced words of length n, for fixed n.

Think of this as a nonbacktracking random walk.

Denote this measure on $\partial\mathbb{F}_2$ by μ_u ("u" for uniform).

One can show that μ_u is shift-invariant, but it's not invariant under the action of $\mathbb{F}_2$. Indeed, $\mu_u([a]) = \frac{1}{4}$, $a^{-1} \cdot [a] = \partial \mathbb{F}_2 \setminus [a^{-1}] = [a] \cup [b] \cup [b^{-1}]$ has measure $\frac{3}{4}$. This is a nice example of a non-pmp action of $\mathbb{F}_2$.

Exercise. Show that $\mathbb{F}_2 \curvearrowright \partial \mathbb{F}_2$ is free μ_u-a.e., where *free* means $\forall \gamma \in \mathbb{F}_2$, γ has no fixed point in $\partial \mathbb{F}$, so μ_u-a.e. free is equiv. to $\mathrm{FixedPt}(\gamma)$ being μ_u-null for each $\gamma \in \Gamma$.